Descent constructions for central extensions of infinite dimensional Lie algebras

Arturo Pianzola*, Daniel Prelat† and Jie Sun

Abstract

We use Galois descent to construct central extensions of twisted forms of split simple Lie algebras over rings. These types of algebras arise naturally in the construction of Extended Affine Lie Algebras. The construction also gives information about the structure of the group of automorphisms of such algebras.

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1 Introduction

Throughout k will denote a field of characteristic 0, and $\mathfrak{g}$ a finite dimensional split simple Lie algebra over k.

Given an associative, unital, commutative k-algebra R, we consider the (in general infinite dimensional) k-Lie algebra $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$. The case of $R = k[t^{\pm 1}]$ arises in the untwisted affine Kac-Moody theory, whereof one knows that the "correct" object to study from the representation point of view is not $\mathfrak{g}_R$ itself, but rather its universal central extension.

In the Kac-Moody case, the universal central extension is one-dimensional. This is not so for the "higher nullity" toroidal algebras corresponding to $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$ with $n > 1$. The universal central extensions are in these cases infinite dimensional, and one knows of many interesting central extensions of $\mathfrak{g}_R$ which are not universal (see [MRY] and [EF]).

In this short note, we look at Lie algebras $\mathcal{L}$ which are twisted forms of $\mathfrak{g}_R$. These algebras appear naturally in the study of Extended Affine Lie Algebras, and present a beautiful bridge between Infinite Dimensional Lie Theory and Galois Cohomology ([AABGP], [ABFP], [P2], and [GP1]). The purpose of this short note is to give natural constructions for central extensions of such algebras by descent methods, and to study their group of automorphisms.

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2 Some generalities on central extensions

Let Ł be a Lie algebra over k, and V a k-space. Any cocycle $P \in Z^2(Ł, V)$, where V is viewed as a trivial Ł-module, leads to a central extension

$$0 \longrightarrow V \longrightarrow Ł_P \xrightarrow{\pi} Ł \longrightarrow 0$$

of Ł by V as follows: As a space $Ł_P = Ł \oplus V$, and the bracket $[\ ,\ ]_P$ on $Ł_P$ is given by

$$[x+u, y+v]_P = [x,y] + P(x,y) \text{ for } x, y \in Ł \text{ and } u, v \in V.$$

The isomorphism class of this extension depends only on the class of P in $H^2(Ł, V)$, and this gives in fact a parametrization of all isomorphism classes of central extensions of Ł by V (see for example [MP] or [We] for details). In this situation, we will henceforth naturally identify Ł and V with subspaces of $Ł_P$.

An automorphism $\theta \in \mathrm{Aut}_k(Ł)$ is said to *lift to* $Ł_P$, if there exists an element $\theta_P \in \mathrm{Aut}_k(Ł_P)$ for which the following diagram commutes.

$$\begin{array}{ccc} Ł_P & \xrightarrow{\pi} & Ł \\ {\scriptstyle\theta_P}\big\downarrow & & {\scriptstyle\theta}\big\downarrow \\ Ł_P & \xrightarrow{\pi} & Ł \end{array}$$

We then say that θ_P is a *lift* of θ.

Remark 2.1 By definition, θ_P stabilizes V; thereof inducing an element of $\mathrm{GL}_k(V)$. By contrast, θ_P *need not* stabilize the subspace Ł of $Ł_P$.

Note that $\theta_P(x) - \theta(x) \in V$ for all $x \in Ł$. Since V lies inside the centre $\mathfrak{z}(Ł_P)$ of $Ł_P$, we get the useful equality

$$\theta_P([x,y]_P) = [\theta(x), \theta(y)]_P \text{ for all } x, y \in Ł. \tag{2.2}$$

Lemma 2.3 *Let $\delta : \mathrm{Hom}_k(Ł, V) \to Z^2(Ł, V)$ be the coboundary map, i.e. $\delta(\gamma)(x,y) = -\gamma([x,y])$. For $\theta \in \mathrm{Aut}_k(Ł)$ and $P \in Z^2(Ł, V)$, the following conditions are equivalent.*

(1) θ lifts to $Ł_P$.

(2) There exists $\gamma \in \mathrm{Hom}_k(Ł, V)$ and $\mu \in \mathrm{GL}_k(V)$ such that $\mu \circ P - P \circ (\theta \times \theta) = \delta(\gamma)$.

In particular, for a lift θ_P of θ to exist, it is necessary and sufficient that there exists $\mu \in \mathrm{GL}_k(V)$ for which, under the natural right action of $\mathrm{GL}_k(V) \times \mathrm{Aut}_k(Ł)$ on $H^2(Ł, V)$, the element (μ, θ) fixes the class $[P]$. If this is the case, the lift θ_P can be chosen so that its restriction to V coincides with μ.

Proof. (1)$\Rightarrow$(2) Let us denote the restriction of θ_P to V by μ. Define $\gamma : Ł \to V$ by $\gamma : x \mapsto \theta_P(x) - \theta(x)$. Then $\theta_P(x+v) = \theta(x) + \gamma(x) + \mu(v)$ for all $x \in Ł$ and $v \in V$.

For all x and y in Ł, we have

$$\begin{aligned}
& (\mu \circ P - P \circ (\theta \times \theta))(x, y) = \mu(P(x, y)) - P(\theta(x), \theta(y)) \\
= \; & \mu([x, y]_P - [x, y]) - ([\theta(x), \theta(y)]_P - [\theta(x), \theta(y)]) \\
= \; & \theta_P([x, y]_P - [x, y]) - [\theta(x), \theta(y)]_P + [\theta(x), \theta(y)] \\
= \; & \theta_P([x, y]_P) - \theta_P([x, y]) - [\theta(x), \theta(y)]_P + [\theta(x), \theta(y)] \\
= \; & \theta_P([x, y]_P) - (\theta([x, y]) + \gamma([x, y])) - [\theta(x), \theta(y)]_P + [\theta(x), \theta(y)] \\
= \; & [\theta(x), \theta(y)]_P - \gamma([x, y]) - [\theta(x), \theta(y)]_P \text{ (by 2.2 above)} \\
= \; & -\gamma([x, y]) = \delta(\gamma)(x, y).
\end{aligned}$$

(2)⇒(1) Define $\theta_P \in \mathrm{End}_k(Ł_P)$ by

$$\theta_P(x + v) = \theta(x) + \gamma(x) + \mu(v) \tag{2.4}$$

for all $x \in \mathcal{L}$ and $v \in V$. Then θ_P is bijective; its inverse being given by $\theta_P^{-1}(x+v) = \theta^{-1}(x) - \mu^{-1}\gamma(\theta^{-1}(x)) + \mu^{-1}(v)$. That θ_P is a Lie algebra homomorphism is straightforward. Indeed,

$$\begin{aligned}
& \theta_P[x + u, y + v]_P = \theta_P([x, y] + P(x, y)) \\
= \; & \theta([x, y]) + \gamma([x, y]) + \mu \circ P(x, y) \\
= \; & \theta([x, y]) + (\mu \circ P - \delta(\gamma))(x, y) \\
= \; & [\theta(x), \theta(y)] + P(\theta(x), \theta(y)) \\
= \; & [\theta(x), \theta(y)]_P = [\theta_P(x + u), \theta_P(y + v)]_P.
\end{aligned}$$

Since the action of $\mathrm{Aut}_k(Ł) \times \mathrm{GL}_k(V)$ on $Z^2(Ł, V)$ in question is given by $P^{(\mu,\theta)} = \mu^{-1} \circ P \circ (\theta \times \theta)$, the final assertion is clear.[1] □

For future use, we recall the following fundamental fact.

Proposition 2.5 *Let Ł be a perfect Lie algebra over k. Then*

(1) There exists a (unique up to isomorphism) universal central extension

$$0 \longrightarrow V \longrightarrow \widehat{Ł} \xrightarrow{\pi} Ł \longrightarrow 0.$$

(2) If Ł is centreless, the centre $\mathfrak{z}(\widehat{Ł})$ of $\widehat{Ł}$ is precisely the kernel V of the projection homomorphism $\pi : \widehat{Ł} \to Ł$ above. Furthermore, the canonical map $Aut_k(\widehat{Ł}) \to Aut_k(Ł)$ is an isomorphism.

Proof. (1) The existence of an initial object in the category of central extensions of Ł is due to Garland [Grl]. (See also [Ne], [MP] and [We] for details).

(2) This result goes back to van der Kallen [vdK]. Other proofs can be found in [Ne] and [P1]. □

[1] This fact is a Lie algebra version of a well known result in group theory. For a much more general discussion of the automorphism group of non-abelian extensions, the reader can consult [Nb] (specially Theorem B2 and its Corollary).

Remark 2.6 Assume Ł is perfect and centreless. We fix once and for all a universal central extension $0 \longrightarrow V \longrightarrow \widehat{Ł} \xrightarrow{\pi} Ł \longrightarrow 0$ (henceforth referred to as *the* universal central extension of Ł). We will find it useful at times to think of this extension as being given by a (fixed in our discussion) "universal" cocycle[2] $\widehat{P}$, i.e. $\widehat{Ł} = Ł_{\widehat{P}} = Ł \oplus V$. The space V is then the centre of $\widehat{Ł}$, and we write $\widehat{Ł} = Ł \oplus \mathfrak{z}(\widehat{Ł})$ to emphasize this point.

Lemma 2.7 *Assume Ł is centreless and perfect, and let $0 \to \mathfrak{z}(\widehat{Ł}) \to \widehat{Ł} \to Ł \to 0$ be its universal central extension. For an automorphism $\theta \in Aut_k(Ł)$, the following conditions are equivalent.*

(1) The lift $\widehat{\theta}$ of θ to $\widehat{Ł}$ acts on the centre of $\widehat{Ł}$ by scalar multiplication, i.e. $\widehat{\theta}|_{\mathfrak{z}(\widehat{Ł})} = \lambda\,\mathrm{Id}$ for some $\lambda \in k^\times$.

(2) θ lifts to every central quotient of $\widehat{Ł}$.

(3) θ lifts uniquely to every central quotient of $\widehat{Ł}$.

(4) θ lifts to every central extension of Ł.

Proof. (1)⇒(2) The lift $\widehat{\theta}$ exists by Proposition 2.5.2. Let $Ł_P$ be a central quotient of $\widehat{Ł}$. Then $Ł_P \simeq \widehat{Ł}/J = (Ł \oplus \mathfrak{z}(\widehat{Ł}))/J \simeq Ł \oplus \mathfrak{z}(\widehat{Ł})/J$. Since $\widehat{\theta}|_{\mathfrak{z}(\widehat{Ł})} = \lambda\,\mathrm{Id}$ for some $\lambda \in k^\times$, we have $\widehat{\theta}(J) \subset J$. So $\widehat{\theta}$ induces an automorphism of $Ł_P$.

(2) ⇒ (3) The point is that $0 \to J \to \widehat{Ł} \to Ł \oplus \mathfrak{z}(\widehat{Ł})/J \to 0$ is a universal central extension of $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$. By Proposition 2.5.2 then, any two lifts of θ to $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$ must coincide (since they both yield $\widehat{\theta}$ when lifted to $\widehat{Ł}$).

(3)⇒(4) There is no loss of generality in assuming that the central extension of Ł is given by a cocycle, i.e. that it is of the form $0 \to V \to Ł_P \to Ł \to 0$ for some $P \in Z^2(Ł, V)$. There exists then a unique Lie algebra homomorphisms $\phi : \widehat{Ł} \to Ł_P$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{z}(\widehat{Ł}) & \longrightarrow & \widehat{Ł} & \longrightarrow & Ł & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi|_{\mathfrak{z}(\widehat{Ł})}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle id_Ł} & & \\ 0 & \longrightarrow & V & \longrightarrow & Ł_P & \longrightarrow & Ł & \longrightarrow & 0 \end{array}$$

Let $J = \ker\phi$. Then $J \subset \mathfrak{z}(\widehat{Ł})$, and the central quotient $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$ corresponds to the cocycle Q obtained by reducing modulo J the universal cocycle $\widehat{P}$ chosen in modeling $\widehat{Ł}$ (see Remark 2.6). We have $Ł_P = Ł \oplus V = \phi(Ł \oplus \mathfrak{z}(\widehat{Ł})/J) \oplus V'$ for some suitable subspace V' of V. Let θ_Q be the unique lift of θ to $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$, which we then transfer, via ϕ, to an automorphism θ'_Q of the subalgebra $\phi(Ł \oplus \mathfrak{z}(\widehat{Ł})/J)$ of $Ł_P$. Then $\theta_P = \theta'_Q + \mathrm{Id}_{V'}$ is a lift of θ to $Ł_P$.

(4)⇒(1) Assume $\widehat{\theta}|_{\mathfrak{z}(\widehat{Ł})} \neq \lambda\,\mathrm{Id}$ for all $\lambda \in k^\times$. Then, there exists a line $J \subset \mathfrak{z}(\widehat{Ł})$ such that $\widehat{\theta}(J) \not\subseteq J$. Consider the central quotient $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$. Let θ_P be a lift of θ to $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$. As pointed out before, $0 \to J \to \widehat{Ł} \to Ł \oplus \mathfrak{z}(\widehat{Ł})/J \to 0$ is the

[2]This cocycle is of course not unique.

universal central extension of $Ł \oplus \mathfrak{z}(\widehat{Ł})/J$. So $\widehat{\theta}$ is also the unique lift of θ_P. This forces $\widehat{\theta}(J) \subset J$, contrary to our assumption. $\square$

3 The case of $\mathfrak{g}_R$

Throughout $\mathfrak{g}$ will denote a finite dimensional split simple Lie algebra over k, and R a commutative, associative, unital k-algebra. We view $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$ as a Lie algebra *over* k (in general infinite dimensional) by means of the unique bracket satisfying

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab \tag{3.8}$$

for all $x, y \in \mathfrak{g}$ and $a, b \in R$. Of course $\mathfrak{g}_R$ is also naturally an R-Lie algebra (which is free of finite rank). It will be at all times clear which of the two structures is being considered.

Let $(\Omega_{R/k}, d_R)$ be the R-module of Kähler differentials of the k-algebra R. When no confusion is possible, we will simply write (Ω_R, d). Following Kassel [Ka], we consider the k-subspace dR of Ω_R, and the corresponding quotient map $^- : \Omega_R \to \Omega_R/dR$. We then have a unique cocycle $\widehat{P} = \widehat{P}_R \in Z^2(\mathfrak{g}_R, \Omega_R/dR)$ satisfying

$$\widehat{P}(x \otimes a, y \otimes b) = (x|\, y)\overline{adb}, \tag{3.9}$$

where $(\,|\,)$ denotes the Killing form of $\mathfrak{g}$.

Let $\widehat{\mathfrak{g}_R}$ be the unique Lie algebra over k with underlying space $\mathfrak{g}_R \oplus \Omega_R/dR$, and bracket satisfying

$$[x \otimes a, y \otimes b]_{\widehat{P}} = [x, y] \otimes ab + (x|\, y)\overline{adb}. \tag{3.10}$$

As the notation suggests,

$$0 \longrightarrow \Omega_R/dR \longrightarrow \widehat{\mathfrak{g}_R} \xrightarrow{\pi} \mathfrak{g}_R \longrightarrow 0$$

is the universal central extension of $\mathfrak{g}_R$.[3]

Proposition 3.11 *Let $\theta \in \mathrm{Aut}_k(\mathfrak{g}_R)$, and let $\widehat{\theta}$ be the unique lift of θ to $\widehat{\mathfrak{g}_R}$ (see Proposition 2.5). If θ is R-linear, then $\widehat{\theta}$ fixes the centre Ω_R/dR of $\widehat{\mathfrak{g}_R}$ pointwise. In particular, every R-linear automorphism of $\mathfrak{g}_R$ lifts to every central extension of $\mathfrak{g}_R$.*

Proof. For future use, we begin by observing that $[x \otimes a, x \otimes b]_{\widehat{P}} = (x|x)\overline{adb}$. Let now $\theta \in \mathrm{Aut}_k(\mathfrak{g}_R)$ be R-linear. Fix $x \in \mathfrak{g}$ such that $(x|\, x) \neq 0$, and write $\theta(x) = \sum_i x_i \otimes a_i$.

[3] There are other different realizations of the universal central extension (see [Ne], [MP] and [We] for details on three other different constructions), but Kassel's model is perfectly suited for our purposes.

Then

$$\begin{aligned} 0 &= [x \otimes ab, x \otimes 1]_{\widehat{P}} = \widehat{\theta}([x \otimes ab, x \otimes 1]_{\widehat{P}}) \\ &= [\theta(x \otimes ab), \theta(x \otimes 1)]_{\widehat{P}} \text{ (by 2.2)} \\ &= [\sum_i x_i \otimes aba_i, \sum_j x_j \otimes a_j]_{\widehat{P}} \\ &= \sum_{i,j} [x_i, x_j] \otimes aba_ia_j + \sum_{i,j} (x_i|x_j)\overline{aba_ida_j}. \end{aligned}$$

Thus

$$\sum_{i,j} (x_i|x_j)\overline{aba_ida_j} = 0 = \sum_{i,j} [x_i, x_j] \otimes aba_ia_j. \tag{3.12}$$

Since θ is R-linear, it leaves invariant the Killing form of the R-Lie algebra $\mathfrak{g}_R$. We thus have

$$(x|x)_{\mathfrak{g}} = (x \otimes 1|x \otimes 1)_{\mathfrak{g}_R} = (\theta(x \otimes 1)|\theta(x \otimes 1))_{\mathfrak{g}_R} = \sum_{i,j} (x_i|x_j)a_ia_j. \tag{3.13}$$

We are now ready to prove the Proposition. By Lemma 2.7, it will suffice to show that $\widehat{\theta}$ fixes Ω_R/dR pointwise. Now,

$$\begin{aligned} \widehat{\theta}((x|x)\overline{adb}) &= \widehat{\theta}([x \otimes a, x \otimes b]_{\widehat{P}}) = [\theta(x \otimes a), \theta(x \otimes b)]_{\widehat{P}} \text{ (by 2.2)} \\ &= [\sum_i x_i \otimes aa_i, \sum_j x_j \otimes ba_j]_{\widehat{P}} \\ &= \sum_{i,j} [x_i, x_j] \otimes aba_ia_j + \sum_{i,j} (x_i|x_j)\overline{aa_idba_j} = \sum_{i,j} (x_i|x_j)\overline{aa_idba_j} \text{ (by 3.12)} \\ &= \sum_{i,j} (x_i|x_j)\overline{aa_ibda_j} + \sum_{i,j} (x_i|x_j)\overline{aa_ia_jdb} = \sum_{i,j} (x_i|x_j)\overline{aa_ia_jdb} \text{ (by 3.12)} \\ &= \overline{\sum_{i,j} (x_i|x_j)a_ia_jadb} = (x|x)\overline{adb} \text{ (by 3.13).} \end{aligned}$$

□

Remark 3.14 Each element $\theta \in \mathrm{Aut}_k(R)$ can naturally be viewed as an automorphism of the Lie algebra $\mathfrak{g}_R$ by acting on the R–coordinates, namely $\theta(x \otimes r) = x \otimes {}^{\theta}r$ for all $x \in \mathfrak{g}$ and $r \in R$. The group $\mathrm{Aut}_k(R)$ acts naturally as well on the space Ω_R/dR, so that ${}^{\theta}(\overline{adb}) = \overline{{}^{\theta}ad^{\theta}b}$. A straightforward calculation shows that the map $\widehat{\theta} \in \mathrm{GL}_k(\widehat{\mathfrak{g}_R})$ defined by $\widehat{\theta}(y + z) = \theta(y) + {}^{\theta}z$ for all $y \in \mathfrak{g}_R$ and $z \in \Omega_R/dR$, is an automorphism of the Lie algebra $\widehat{\mathfrak{g}_R}$. Thus $\widehat{\theta}$ is the unique lift of θ to $\widehat{\mathfrak{g}_R}$ prescribed by Proposition 2.5.2. Note that $\widehat{\theta}$ stabilizes the subspace $\mathfrak{g}_R$.

We now make some general observations about the automorphisms of $\mathfrak{g}_R$ that lift to a given central quotient of $\widehat{\mathfrak{g}_R}$. Without loss of generality, we assume that

the central quotient at hand is of the form $(\mathfrak{g}_R)_P$ for some $P \in Z^2(\mathfrak{g}_R, V)$. Let $\theta \in \mathrm{Aut}_k(\mathfrak{g}_R)$. Since $\widehat{\mathfrak{g}_R}$ is the universal central extension of its central quotients, the lift θ_P, if it exists, is unique (Lemma 2.7). We have $\mathrm{Aut}_k(\mathfrak{g}_R) = \mathrm{Aut}_R(\mathfrak{g}_R) \rtimes \mathrm{Aut}_k(R)$ ([ABP] Lemma 4.4. See also [BN] Corollary 2.28). By Proposition 3.11 all elements of $\mathrm{Aut}_R(\mathfrak{g}_R)$ do lift, so the problem reduces to understanding which $\theta \in \mathrm{Aut}_k(R)$ admit a lift θ_P to $(\mathfrak{g}_R)_P$. Since $\widehat{\theta}$ stabilizes $\mathfrak{g}_R$, the linear map γ of Lemma 2.3 vanishes.[4] We conclude that θ_P exists if and only if there exists a linear automorphism $\mu \in \mathrm{GL}_k(V)$ such that $P = P^{(\mu,\theta)}$.

Example 3.15 Let $k = \mathbb{C}$ and $R = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. Fix $\zeta \in \mathbb{C}$, and consider the one dimensional central extension $\text{Ł}_{P_\zeta} = \mathfrak{g}_R \oplus \mathbb{C}c$, with cocycle P_ζ given by $P_\zeta(x \otimes t_1^{m_1} t_2^{m_2}, y \otimes t_1^{n_1} t_2^{n_2}) = (x|\, y)(m_1 + \zeta m_2)\delta_{m_1+n_1,0}\delta_{m_2+n_2,0}c$ (see [EF]). We illustrate how our methods can be used to describe the group of automorphisms of this algebra.

As explained in the previous Remark, $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g}_R) = \mathrm{Aut}_R(\mathfrak{g}_R) \rtimes \mathrm{Aut}_{\mathbb{C}}(R)$, all R-linear automorphisms of $\mathfrak{g}_R$ lift to Ł_{P_ζ}, and we are down to understanding which elements of $\mathrm{Aut}_{\mathbb{C}}(R)$ can be lifted to $\mathrm{Aut}_{\mathbb{C}}(\text{Ł}_{P_\zeta})$.

Each $\theta \in \mathrm{Aut}_{\mathbb{C}}(R)$ is given by $\theta(t_1) = \lambda_1 t_1^{p_1} t_2^{p_2}$ and $\theta(t_2) = \lambda_2 t_1^{q_1} t_2^{q_2}$ for some $\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, and $\lambda_1, \lambda_2 \in \mathbb{C}^\times$. The natural copy of the torus $\mathbf{T} = \mathbb{C}^\times \times \mathbb{C}^\times$ inside $\mathrm{Aut}_{\mathbb{C}}(R)$ clearly lifts to $\mathrm{Aut}_{\mathbb{C}}(\text{Ł}_{P_\zeta})$. We are thus left with describing the group $\mathrm{GL}_2(\mathbb{Z})_\zeta$ consisting of elements of $\mathrm{GL}_2(\mathbb{Z})$ that admit a lift to $\mathrm{Aut}_{\mathbb{C}}(\text{Ł}_{P_\zeta})$. Then $\mathrm{Aut}_{\mathbb{C}}(\text{Ł}_{P_\zeta}) \simeq \mathrm{Aut}_R(\mathfrak{g}_R) \rtimes \big((\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \mathrm{GL}_2(\mathbb{Z})_\zeta\big)$. Note also that since $\mathrm{Pic}(R) = 1$, the structure of the group $\mathrm{Aut}_R(\mathfrak{g}_R)$ is very well understood [P1].

The $\mathrm{GL}_2(\mathbb{Z})_\zeta$ form an interesting 1-parameter family of subgroups of $\mathrm{GL}_2(\mathbb{Z})$ that we now describe. Let $\theta \in \mathrm{GL}_2(\mathbb{Z})$. Then by Lemma 2.3, $\theta \in \mathrm{GL}_2(\mathbb{Z})_\zeta$ if and only if there exists $\gamma : \mathfrak{g}_R \to \mathbb{C}c$ and $\mu \in \mathrm{GL}_{\mathbb{C}}(\mathbb{C}c) \simeq \mathbb{C}^\times$, such that

$$\gamma([x \otimes a, y \otimes b]) = (P \circ (\theta \times \theta) - \mu P)(x \otimes a, y \otimes b) \tag{3.16}$$

for all $x, y \in \mathfrak{g}$, and all $a, b \in R$ (in fact $\gamma = 0$, as explained in Remark 3.14) . Choose $x, y \in \mathfrak{g}$ with $(x|\, y) \neq 0$. Since in $\mathfrak{g}_R$ we have $[x \otimes 1, y \otimes 1] = [x \otimes t_1, y \otimes t_1^{-1}] = [x \otimes t_2, y \otimes t_2^{-1}]$, a straightforward computation based on (3.16) yields

$$\theta \begin{pmatrix} 1 \\ \zeta \end{pmatrix} := \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} = \mu \begin{pmatrix} 1 \\ \zeta \end{pmatrix}. \tag{3.17}$$

The group $\mathrm{GL}_2(\mathbb{Z})_\zeta$ could thus be trivial, finite, or even infinite, depending on some arithmetical properties of the number ζ. For example if $\zeta^2 = -1$, then $\mathrm{GL}_2(\mathbb{Z})_\zeta$ is a cyclic group of order 4, generated by the element σ for which $\sigma(t_1) = t_2$ and $\sigma(t_2) = t_1^{-1}$.

The one-parameter family $\mathrm{GL}_2(\mathbb{Z})_\zeta$ has the following interesting geometric interpretation (which was suggested to us by the referee). By the universal nature of $\widehat{\mathfrak{g}_R} =$

[4]This holds for every central extension of $\mathfrak{g}_R$, and not just central quotients of $\widehat{\mathfrak{g}_R}$. We leave the details of this general case to the reader.

$\mathfrak{g}_R \oplus \Omega_R/dR$, we can identify the space $Z^2(\mathfrak{g}_R, \mathbb{C})$ of cocycles with $\mathrm{Hom}(\Omega_R/dR, \mathbb{C}) = (\Omega_R/dR)^*$. Furthermore, this identification is compatible with the respective actions of the group $\mathrm{Aut}_k(R)$.

The action of the torus $\mathbf{T}$ on Ω_R/dR is diagonalizable, and the fixed point space $V = (\Omega_R/dR)^{\mathbf{T}}$ is two dimensional with basis $\{\overline{t_1^{-1}dt_1}, \overline{t_2^{-1}dt_2}\}$. The cocycle P_ζ is $\mathbf{T}$–invariant, and corresponds to the linear function $F_\zeta \in (\Omega_R/dR)^*$ which maps $\overline{t_1^{-1}dt_1} \mapsto 1$, $\overline{t_2^{-1}dt_2} \mapsto \zeta$, and vanishes on all other weight spaces of $\mathbf{T}$ on Ω_R/dR. We can thus identify F_ζ with an element of V^*. The action of $\mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{Aut}_{\mathbb{C}}(R)$ on Ω_R/dR stabilizes V, and is nothing but left multiplication with respect to the chosen basis above.

Let $\theta \in \mathrm{GL}_2(\mathbb{Z})$. Then θ lifts to $Ł_{P_\zeta}$ if and only if $P_\zeta = P_\zeta^{(\mu,\theta)}$ for some $\mu \in \mathbb{C}^\times$ (Remark 3.14). With the above interpretation, this is equivalent to θ stabilizing the line $\mathbb{C}F_\zeta \subset V^*$; which is precisely equation (3.17) (after one identifies V with V^* via our choice of basis).

4 The case of twisted forms of $\mathfrak{g}_R$

We now turn our attention to forms of $\mathfrak{g}_R$ for the flat topology of R, i.e. we look at R-Lie algebras Ł for which there exists a faithfully flat and finitely presented extension S/R for which

$$Ł \otimes_R S \simeq \mathfrak{g}_R \otimes_R S \simeq \mathfrak{g} \otimes_k S, \tag{4.18}$$

where the above are isomorphisms of S-Lie algebras.

Let $\mathbf{Aut}(\mathfrak{g})$ be the k-algebraic group of automorphisms of $\mathfrak{g}$. The R-group $\mathbf{Aut}(\mathfrak{g})_R$ obtained by base change is clearly isomorphic to $\mathbf{Aut}(\mathfrak{g}_R)$. It is an affine, smooth, and finitely presented group scheme over R whose functor of points is given by

$$\mathbf{Aut}(\mathfrak{g}_R)(S) = \mathrm{Aut}_S(\mathfrak{g}_R \otimes_R S) \simeq \mathrm{Aut}_S(\mathfrak{g} \otimes_k S). \tag{4.19}$$

By Grothendieck's theory of descent (see [Mln] and [SGA3]), we have a natural bijective map

$$\text{Isomorphism classes of forms of } \mathfrak{g}_R \longleftrightarrow H^1_{\acute{e}t}\big(R, \mathbf{Aut}(\mathfrak{g}_R)\big). \tag{4.20}$$

In the case when $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$, the class of algebras on the left plays an important role in modern infinite dimensional Lie theory. For $n = 1$ the forms in question are nothing but the affine Kac-Moody algebras (derived modulo their centres. See [P2]). For general n, these algebras yield all the centerless cores of Extended Affine Lie Algebras (EALA) which are finitely generated over their centroids [ABFP].

Neher has shown how to "build" EALAs out of their centerless cores (in particular, his methods yield all central extensions of such cores) [Ne]. We now illustrate how to naturally build central extensions for twisted forms of $\mathfrak{g}_R$ by descent considerations. In the case when the descent data corresponds to an EALA, the resulting algebra is the universal central extension of the corresponding centreless core.

Henceforth S/R will be *finite* Galois with Galois group G (see [KO]). We will assume that our Ł is split by such an extension[5], i.e.

$$Ł \otimes_R S \simeq \mathfrak{g} \otimes_k S \tag{4.21}$$

as S-Lie algebras. The descent data corresponding to Ł, which a priori is an element of $\mathbf{Aut}(\mathfrak{g})(S \otimes_R S)$, can now be thought as being given by a cocycle $u \in Z^1(G, \mathrm{Aut}_S(\mathfrak{g}_S))$ (usual non-abelian Galois cohomology), where the group G acts on $\mathrm{Aut}_S(\mathfrak{g}_S) = \mathrm{Aut}_S(\mathfrak{g} \otimes_k S)$ via ${}^g\theta = (1 \otimes g) \circ \theta \circ (1 \otimes g^{-1})$.

As above, we let (Ω_S, d) be the module of Kähler differentials of S/k. The Galois group G acts naturally both on Ω_S and on the quotient k-space Ω_S/dS, in such way that ${}^g(\overline{sdt}) = \overline{{}^g s d {}^g t}$. This leads to an action of G on $\widehat{\mathfrak{g}_S}$ for which

$${}^g(x \otimes s + z) = x \otimes {}^g s + {}^g z$$

for all $x \in \mathfrak{g}$, $s \in S$, $z \in \Omega_S/dS$, and $g \in G$. One verifies immediately that the resulting maps are automorphisms of the k-Lie algebras $\widehat{\mathfrak{g}_S}$. Indeed, ${}^g[x_1 \otimes s_1 + z_1, x_2 \otimes s_2 + z_2]_{\widehat{\mathfrak{g}_S}} ={}^g ([x_1, x_2] \otimes s_1 s_2 + (x_1|\, x_2)\overline{s_1 ds_2}) = [x_1, x_2] \otimes {}^g s_1 {}^g s_2 + (x_1|\, x_2)\overline{{}^g s_1 d {}^g s_2} = [x_1 \otimes {}^g s_1 + {}^g z_1, x_2 \otimes {}^g s_2 + {}^g z_2]_{\widehat{\mathfrak{g}_S}}$. Accordingly, we henceforth identify G with a subgroup of $\mathrm{Aut}_k(\widehat{\mathfrak{g}_S})$, and let then G act on $\mathrm{Aut}_k(\widehat{\mathfrak{g}_S})$ by conjugation, i.e. ${}^g\theta = g\theta g^{-1}$.

Proposition 4.22 *Let $u = (u_g)_{g\in G}$ be a cocycle in $Z^1\big(G, \mathrm{Aut}_S(\mathfrak{g}_S)\big)$. Then*

(1) $\widehat{u} = (\widehat{u}_g)_{g\in G}$ is a cocycle in $Z^1(G, \mathrm{Aut}_k(\widehat{\mathfrak{g}_S}))$.

(2) $Ł_{\widehat{u}} = \{x \in \widehat{\mathfrak{g}_S} :\ \widehat{u}_g {}^g x = x \text{ for all}\, g \in G\}$ is a central extension of the descended algebra $Ł_u$ corresponding to u.

(3) There exist canonical isomorphisms $\mathfrak{z}(Ł_{\widehat{u}}) \simeq (\Omega_S/dS)^G \simeq \Omega_R/dR$.

Proof. (1) $\widehat{u}_{g_1} {}^{g_1}\widehat{u}_{g_2}$ is clearly a lift to $\mathrm{Aut}_k(\widehat{\mathfrak{g}_S})$ of $u_{g_1} {}^{g_1} u_{g_2} = u_{g_1 g_2} \in \mathrm{Aut}_k(\mathfrak{g}_S)$. By uniqueness (Proposition 2.5.2), we have $\widehat{u}_{g_1 g_2} = \widehat{u}_{g_1} {}^{g_1}\widehat{u}_{g_2}$.

(2) It is clear that $Ł_{\widehat{u}}$ is a k-subalgebra of $\widehat{\mathfrak{g}_S}$. Moreover if $x \in \mathfrak{g}_S$ and $\omega \in \Omega_S$ are such that $x + \overline{\omega} \in Ł_{\widehat{u}}$, then

$$x + \overline{\omega} = \widehat{u}_g {}^g(x + \overline{\omega}) = \widehat{u}_g({}^g x +{}^g \overline{\omega}) = u_g {}^g x + (\widehat{u}_g - u_g)({}^g x) +{}^g \overline{\omega}$$

(this last equality by Proposition 3.11 applied to $\mathfrak{g}_S$). Since $(\widehat{u}_g - u_g)(\mathfrak{g}_S) \subset \Omega_S/dS$ we get $x = u_g {}^g x$, hence that $x \in Ł_u$. Thus $\pi(Ł_{\widehat{u}}) \subset Ł_u$, where

$$0 \longrightarrow \Omega_S/dS \longrightarrow \widehat{\mathfrak{g}_S} \xrightarrow{\pi} \mathfrak{g}_S \longrightarrow 0$$

as above, is the universal central extension of $\mathfrak{g}_S$. Since the kernel of $\pi|_{Ł_{\widehat{u}}} : Ł_{\widehat{u}} \to Ł_u$ is visibly central, the only delicate point is to show that $\pi(Ł_{\widehat{u}}) = Ł_u$.

Fix $x \in Ł_u$. We must show the existence of some $z \in \Omega_S/dS$ for which $x + z \in Ł_{\widehat{u}}$. For each $g \in G$, we have $\widehat{u}_g {}^g x = x + x_g$ for some $x_g \in \Omega_S/dS$. We claim that the map $g \mapsto x_g$ is a cocycle in $Z^1(G, \Omega_S/dS)$. Indeed,

$$x + x_{g_1 g_2} = \widehat{u}_{g_1 g_2} {}^{g_1 g_2} x = \widehat{u}_{g_1} {}^{g_1}\widehat{u}_{g_2} {}^{g_1 g_2} x = \widehat{u}_{g_1} g_1 \widehat{u}_{g_2} {}^{g_2} x = \widehat{u}_{g_1} {}^{g_1}(x + x_{g_2}) = x + x_{g_1} + {}^{g_1} x_{g_2},$$

[5] By the Isotriviality Theorem of [GP1], this assumption is superflous for $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$.

and the claim follows. Given that G is finite and k is of characteristic 0, we have $H^1(G, \Omega_S/dS) = 0$. Thus, there exists $z \in \Omega_S/dS$ such that $x_g = z - {}^g z$ for all $g \in G$. Then $\widehat{u}_g{}^g(x+z) = x + x_g + {}^g z = x + z$, so that $x + z \in Ł_{\widehat{u}}$ as desired.

(3) Because $Ł_u$ is centerless (see Remark 4.24 below), the centre of $Ł_{\widehat{u}}$ lies inside Ω_S/dS, hence inside $(\Omega_S/dS)^G$ by the definition of $Ł_{\widehat{u}}$ together with Proposition 3.11 applied to $\mathfrak{g}_S$. Thus, under the canonical identification of Ω_S/dS with a subspace of $Ł_{\widehat{u}}$, we have $\mathfrak{z}(Ł_{\widehat{u}}) = (\Omega_S/dS)^G$.

On the other hand $\Omega_S/dS \simeq HC_1(S)$, where this last is the cyclic homology of S/k. The group G acts naturally on the $HC_1(S)$, and the canonical isomorphism $\Omega_S/dS \to HC_1(S)$ is G-equivariant. Since S/R is Galois, naïve descent holds for cyclic homology ([WG] proposition 3.2). We thus have

$$(\Omega_S/dS)^G \simeq HC_1(S)^G \simeq HC_1(S^G) \simeq HC_1(R) \simeq \Omega_R/dR.$$

$\square$

Proposition 4.23 *With the above notation, the following conditions are equivalent.*

(1) $Ł_{\widehat{u}} = Ł_u \oplus \Omega_R/dR$ *and* $Ł_u$ *is stable under the action of the Galois group* G*.*

(2) $\widehat{u}_g(Ł_u) \subset Ł_u$ *for all* $g \in G$*.*

If these conditions hold, then every $\theta \in Aut_R(Ł_u)$ *lifts to an automorphism* $\widehat{\theta}$ *of* $Ł_{\widehat{u}}$ *that fixes the centre of* $Ł_{\widehat{u}}$ *pointwise.*

Proof. (1)⇒(2) Let $x \in Ł_u$. By assumption we have ${}^{g^{-1}}x \in Ł_u \subset Ł_{\widehat{u}}$ for all $g \in G$. Thus $\widehat{u}_g(x) = \widehat{u}_g{}^g({}^{g^{-1}}x) = {}^{g^{-1}}x \in Ł_u$.

(2)⇒(1) Let $x \in Ł_u$. From the assumption $\widehat{u}_g(Ł_u) \subset Ł_u$ we obtain $\widehat{u}_g(x) = u_g(x) \in Ł_u$ for all $g \in G$. Thus

$${}^{g^{-1}}x = u_{g^{-1}g}({}^{g^{-1}}x) = u_{g^{-1}}{}^{g^{-1}}u_g({}^{g^{-1}}x) = u_{g^{-1}}{}^{g^{-1}}(u_g(x)) = u_g(x) \in Ł_u.$$

This shows that $Ł_u$ is G–stable. Now ${}^g x \in Ł_u$ yields $\widehat{u}_g({}^g x) = u_g({}^g x) = x$. Thus implies $Ł_u \subset Ł_{\widehat{u}}$. By Proposition 3.11 $\widehat{u}_g$ fixes Ω_S/dS pointwise, so we have $\Omega_R/dR \subset Ł_{\widehat{u}}$. Thus $Ł_u \oplus \Omega_R/dR \subset Ł_{\widehat{u}}$. Finally, if $\widehat{x} \in Ł_{\widehat{u}}$ and we write $\widehat{x} = x + z$ with $x \in Ł_u$ and $z \in \Omega_S/dS$ according to Proposition 4.22.2, then $\widehat{x} - x = z \in Ł_{\widehat{u}}$. Thus ${}^g z = \widehat{u}_g{}^g z = z$. This shows that $z \in \Omega_R/dR$ (see Proposition 4.22.3).

As for the final assertion, let θ_S be the unique S-Lie automorphism of $\mathfrak{g}_S$ whose restriction to $Ł_u$ coincides with θ. Let $\widehat{\theta_S}$ be the lift of θ_S to $\widehat{\mathfrak{g}_S}$. We claim that $\widehat{\theta_S}$ stabilizes $Ł_{\widehat{u}} = Ł_u \oplus \Omega_R/dR$. By Proposition 3.11 $\widehat{\theta_S}$ fixes Ω_R/dR pointwise. Let $x \in Ł_u$. Since $Ł_u$ is perfect (see Remark 4.24 below), we can write $x = \sum_i [x_i, y_i]_{\mathfrak{g}_S}$ for some $x_i, y_i \in Ł_u$. Thus $x = \sum_i [x_i, y_i]_{\widehat{\mathfrak{g}_S}} - z$ for some $z \in \Omega_R/dR$. Then $\widehat{\theta_S}(x) = \sum_i [\theta(x_i), \theta(y_i)]_{\widehat{\mathfrak{g}_S}} - z \in [Ł_u, Ł_u]_{\widehat{\mathfrak{g}_S}} + \Omega_R/dR \subset Ł_u \oplus \Omega_R/dR = Ł_{\widehat{u}}$. $\square$

Remark 4.24 Let Ł be a twisted form of $\mathfrak{g}_R$ in the sense of (4.18) above. By faithfully flat descent considerations, Ł is centreless. Indeed, the centre $\mathfrak{z}(Ł) \subset Ł$ is an R-submodule of Ł. Since S/R is faithfully flat, the map $\mathfrak{z}(Ł) \otimes_R S \to Ł \otimes_R S \simeq \mathfrak{g} \otimes S$ is injective. Clearly the image of $\mathfrak{z}(Ł) \otimes_R S$ under this map lies inside the centre of

$\mathfrak{g} \otimes S$, which is trivial (as one easily sees by considering a k-basis of S, and using the fact that $\mathfrak{z}(\mathfrak{g}) = 0$). Thus $\mathfrak{z}(\text{Ł}) \otimes_R S = 0$, and therefore $\mathfrak{z}(\text{Ł}) = 0$ again by faithfull flatness. Similarly descent considerations (see §5.1 and §5.2 of [GP2] for details) show that Ł is perfect, and that the centroid of Ł, both as an R and k-Lie algebra, coincides with R (acting faithfully on Ł via the module structure).

Assume now that Ł is split by a finite Galois extension S/R. Let G be the Galois group of S/R. Then $\text{Ł} \simeq \text{Ł}_u$ for some cocycle $u = (u_g)_{g \in G}$ as above. The R-group $\mathbf{Aut}(\text{Ł}_u)$ is a twisted form of $\mathbf{Aut}(\mathfrak{g}_R)$ (§5.4 of [GP2]) , in particular affine, smooth, and finitely presented. We have $\text{Aut}_R(\text{Ł}_u) = \mathbf{Aut}(\text{Ł}_u)(R)$. Every automorphism of Ł_u as a k-Lie algebra induces an automorphism of its centroid. By identifying now the centroid of Ł_u with R as explained above, we obtain the following useful exact sequence of groups

$$1 \to \text{Aut}_R(\text{Ł}_u) \to \text{Aut}_k(\text{Ł}_u) \to \text{Aut}_k(R). \tag{4.25}$$

If moreover the descent data for Ł_u falls under the assumption of Proposition 4.23 (which includes the EALA case[6]), then one also has a very good understanding of the automorphism group of $\text{Ł}_{\widehat{u}}$. This group will undoubtedly play a role in any future work dealing with conjugacy questions for Extended Affine Lie Algebras (see [P3] and [P1] for the toroidal case).

Finally, we observe that since Ł_u above is perfect, it admits a universal central extension $\widehat{\text{Ł}_u}$. By Proposition 4.22, there exists a canonical map $\widehat{\text{Ł}_u} \to \text{Ł}_{\widehat{u}}$. In the case of descent data arising from EALAs, this map is an isomorphism (work in progress of Neher, given that $\text{Ł}_{\widehat{u}} = \text{Ł}_u \oplus \Omega_R/dR$ by [ABFP] *loc .cit.* as explained above). What happens in general however, remains an open problem.

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[6]The crucial point is that for EALAs, the algebra Ł_u may be assumed to be a multiloop algebra ([ABFP], corollary 8.3.5). By a general fact about the nature of multiloop algebras as forms (see [P2] for loop algebras, and [GP2] §6 in general), the cocycle u is a group homomorphism $u : G \to \text{Aut}_k(\mathfrak{g})$. In particular, u is constant (i.e. it has trivial Galois action). The multiloop algebra Ł_u has then a basis consisting of eigenvectors of the u_g's, and therefore the second equivalent condition of Proposition 4.23 holds.

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Arturo Pianzola and Jie Sun, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1 CANADA.

Daniel Prelat, Departamento de Matemática, Universidad CAECE, Avenida de Mayo 866, Buenos Aires ARGENTINA.